International Journal of Theoretical Physics, Vol. 23, No. 11, 1984

Relativity of Topology and Dynamics

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Received February 24, 1984

Recent developments in quantum set theory are used to formulate a program for quantum topological physics. The world is represented in a Hilbert space whose psi vectors represent abstract complexes generated from the null set by one bracket operator and the usual Grassmann (or Clifford) product. Such a theory may be more basic than field theory, in that it may generate its own natural topology, time, kinematics and dynamics, without benefit of an absolute time-space dimension, topology, or Hamiltonian. For example there is a natural expression for the quantum gravitational field in terms of quantum topological operators. In such a theory the usual spectrum of possible dimensions describes only one of an indefinite hierarchy of levels, each with a similar spectrum, describing nonspatial infrastructure. While c simplices have no continuous symmetry, the q simplex has an orthogonal group $O(m, n)$. Because quantum theory cannot take the universe as physical system, we propose a "third relativity:" *The division between observer and observed is arbitrary.* Then it is wrong to ask for "the" topology and dynamics of a system, in the same sense that it is wrong to ask for the "the" psi vectors of a system; topology and dynamics, like psi vectors, are not absolute but relative to the observer.

1. THE MANY-STORIED TOWER

The question we ask is still: What is the simplest viable theory of a quantum world? The answer to which we now have come is: That the world is a quantum set, and therefore a quantum simplicial complex. A simplicial complex, we recall, is a set of simplices, and a simplex is a polyhedron with the minimum number of vertices for its dimension. The simplices with which we build our theory are not "concrete" simplices, polyhedra in an affine space, but "abstract," completely defined by the set consisting of their vertices, which we build from nothing using the operations of set theory.

0020-7748/84/1100-1065$03.50/0

The world is elegantly expressed as a complex of concrete polyhedra by Regge (1961), of abstract cells by Susskind (1977), and of concrete simplices by Christ et al. (1982) and Lee (1983). These programs assume commuting coordinates and both Regge and Lee put a classical time space continuum into their polyhedra, where clocks and metersticks cannot go. These vestiges of the continuous we consider as provisional intermediaries on the way to a deeper theory. At the same time we do not create our theory from a state of pristine ignorance. We assume that present physical theory works as well as it does because it has a structure that is correct in important ways, and we demand that our theory have a correspondence principle governing the transition from the discrete to the continuous, a singular limit in which the chronon or fundamental time $\hat{\tau} \to 0$.

In this paper we apply this correspondence principle to the vertical or hierarchic structure of physics, as well as to the more familiar horizontal time-space structure, and we are lead to postulate that the vertices of the world complex are fermi, and that the simplices have an internal quantum geometry already defined by the vertex algebra. Moreover, the topological degrees of freedom of our quantum simplicial complex, with no additional metrical variables, already define a quantum gravitational field.

By the vertical structure of physics we mean the following. If we follow the existing theories of physics to their mathematical foundations, we always find that present theories, even those called unified, are hierarchic towers of several heterogeneous stories of increasing abstraction and depth:

Story 6. Empirics. This would once have been a point at the top of the tower, representing the values of all the coordinates of the world; reality.

Story 5. Dynamics. If we do not know the values of all the coordinates, we may still know the law of nature; nowadays, Hamiltonians, Lagrangians, or action principles.

Story 4. Kinematics. If we do not know the law, we may still know the degrees of freedom; nowadays, gauge fields, such as the electromagnetic and its weak and strong generalizations, and their sources.

Story 3. Time. If we do not know what the fields are, we may still know their time space geometry: the gravitational pseudometric tensor g of Einstein. Some theories have no floor between stories 3 and 4.

Story 2. Topology. If we do not know time from space we may still know what is connected to what: the open set structure and the differential. In principle this might be two stories; we will not need such detail.

Story 1. Set Theory. If we do not know how the points are connected, we may still know them individually and in sets. In *Principia Mathematica*, set theory is the theory of the membership relation; we renounce this concept of set theory soon.

Story 0. Predicate Logic. If we do not know the points of the theory, we may still know the logical laws obeyed by predicates about these points: the algebra of the logical operations *and*, *or*, *not*. These laws are defined by the algebra (a lattice) of subsets of phase space for classical physics, and of the Hilbert space projection operators (henceforth: *projectors*) for quantum systems.

Quantization on first encounter seems a penthouse on this tower, or a new story; but Von Neumann (1932) points out that quantization transforms story 0, from a logic without complementarity to a complementarity logic. It is important for our work that complementarity leads to a second relativity, deeper than the first, with unitary transformations of a Hilbert space basis instead of point transformations of a manifold as the relativity group (Davis, 1977). Thus the construction has returned to ground level. The quantum field theories of today, with their classical foundations, are incoherent, grafts of quantum limbs onto classical bodies, and may therefore be provisional, a hybrid classical–quantum (cq) stage in the development of physics from totally classical (c) to wholly quantum (q). To preserve correspondence from q to cq to c and to construct a coherent theory, we should rebuild the tower on a base of q predicate algebra, with a q set theory, q topology, q time, and so forth to the top.

A major challenge for this program is a basic theory of the gauge structure and particles of nature. While gauge theory began as a theory of the electromagnetic field, with the most simple gauge transformation possible, it has evolved into a much richer theory, involving all the interactions, including gravity. Our working hypothesis is a "plexus principle": The observed gauge structure is too complex to be basic and arises from a basic "plexus," or q topology (defined below), in a c continuum limit; somewhat as the Burger's vectors of the solid state, which describe a gauge field appropriate to crystals, arise from the topology of crystals. This is simply the only natural origin for gauge with which we are acquainted. [Duerr and Saller (1981) and Veneziano (1983) also consider gauge fields to be nonbasic and composed of fermions. Riemann already declared that the metrical properties of discrete manifolds were determined by their topology.]

The plexus principle suggests that the right topology provides its own kinematics, and that many fields will appear in the continuum approximation, not one unified field, when many kinds of defect are possible in the plexus.

Each of the following sections deals with the reconstruction of one story of this tower, starting at the base, until the view from story 3 invites us to sharpen our goal and seek a "basic" theory (Section 5).

Only suggestions of the highest stories are given (Sections 6 and 7). We find, however, that in a basic theory there is no room for an absolute dynamical law, such as a Lagrangian for the universe. There must always be a coordinating system (CS) and a system coordinated (SC). The unity of the universe is expressed in physics not by the absence of such a cut but by its arbitrariness. We call this the *third relativity*. A relative dynamical law says that we, a certain CS, can (or cannot) carry out a certain process on the SC; for example, a certain initial determination $\langle p|$ of photon polarization followed by a certain final one $|p'\langle$. An absolute dynamical law says that a certain process can (or cannot) go on in the universe, speaking from an absolute point of view that is usually unattributed or deified. Measurement theory in some part arises from a wish to derive relative from absolute law. We propose that this wish is futile, and that for law as for time and truth, the actual relations between CS's preclude the reduction of the multitude of relatives to one absolute. By relativizing the base, we relativize the whole tower.

There is a curiously Leibnizian flavor to the theory that emerges from this construction, as though mentioning monads had called up a ghost. It is common to distinguish between the external world of an elementary particle, with coordinates *txyz*, and its internal world, with coordinates like flavor and nucleon number. In our theory, however, every possible world is the internal world of another possible world. Albeit in this weakened sense, the monad mirrors within itself the whole world. As a result the modes of description (topological, geometric, etc.) ordinarily used for the external world now have application at an indefinite number of deeper levels of hierarchy.

Therefore our theory supplements the usual time space dimensions in two ways. First, the basic simplices of the world may have extra dimensions (on the same level as the usual four of time and space), and second, the vertices of each simplex may themselves be simplices (of a deeper level than the usual).

A c complex has sharp points, breaking Lorentz invariance. It is reasonable to suppose that if the world were a c simplicial complex with a scale of time $\hat{\tau}$ for its edges, then violations of angular momentum conservation (among others) would appear at energies $1/\hat{\tau}$. Since such violations are not seen, some suppose that the continuum approximation has been validated to the highest energies we make today.

The q simplex is Lorentz invariant. Since the q simplex has so much symmetry, it seems likely that q topology postpones violations to energies much greater than $1/\hat{\tau}$. Therefore the chronon might be closer to Heisenberg's fundamental scale (1 fm/c) than to Planck's. Lee and collaborators reach the same conclusion by imbedding the simplex in Minkowski time space and averaging over all possible positions and orientations, thereby recovering symmetry.

Now horizontal, one-level theories of companion dimensions like Kaluza–Klein require much smaller times than 1 fm/c for their new dimensions. Therefore we should also explore whether the internal gauge degrees of freedom might arise from levels below time space. Electric charge, for example, may be an ancestor rather than a sibling of energy. It seems possible to explain the weakness of gravity for $\hat{\tau}$ of Heisenberg size rather than to explain the tiny spacing of the mass spectrum for $\hat{\tau}$ of Planck size, when particles are to monads much as galaxies are to atoms.

From the viewpoint of unified field theory, the deeper levels are best ignored, for they raise too many questions. Each level of infrastructure would have to be provided with its own laws. The horizontal road of Kaluza–Klein theory is then the one to take. But from the viewpoint of q set theory, both are valid options, and both might occur in nature, as far as we now know.

In an earlier work (Finkelstein, 1969), each time-space point is given a Clifford algebra of spin-1/2 operators, so that, in Weizsäcker's phrase, the world is a pattern of q binary decisions (between unspecified alternatives, which may be called up or down). The algebras at different points commute, in the way commonly assumed for independent variables. In the present work, there is again a Clifford algebra at each point, though not always the Pauli spin algebra, and operators at different points anticommute. The world is still a pattern of binary q decisions (now specified: to be, or not to be) but they are fermi. Now the binary nature of these decisions has nothing to do with the dimensionality of time space, which changes from place to place.

2. QUANTUM SET THEORY

Where the logic of classical physics represents predicates about the system under study by subclasses of a sample space, called phase space, the logic of current q physics uses subspaces of a Hilbert space, or the projection operators thereon.

As far as the general principles of c logic are concerned, one sample space is as good as another; these principles are no help in choosing the sample space. Nor do they distinguish between points in the sample space: The group of each such predicate algebra is the permutation group of its sample space, and this acts transitively. But then set theory provides a unique sample space, the class of all sets, and provides additional structure that totally reduces the symmetry group (to 1). Classical mathematical physics has no difficulty working within this sample space. The hypothesis that the world is a set seems to eliminate nothing. Classical set theory is universal in a certain sense.

In fact, the world seems *not* to be a c set, since its predicates do not even form a c predicate algebra. The possibility remains, however, that it is a "q set," a concept which we formulate now by analogy.

As far as the general principles of q logic are concerned, one Hilbert space is as good as another; q predicate algebra is no help in choosing the Hilbert space. Nor does it distinguish between Hilbert vectors: The resulting predicate algebra is invariant under the unitary group of its Hilbert space.

This situation is intolerable for physics. We require a definite Hilbert space, and we require sufficient further structure to describe the physical difference that we experience between the processes represented by different transformations of this space.

It is plausible by analogy with c set theory that q set theory have a unique Hilbert space, additional structure totally reducing the unitary group, and a universality similar to c set theory. A correct q set theory might satisfy the physical need for a unique Hilbert space and a totally broken unitary group, as well as the desire for a coherent theory. This provides further motivation for the following constructions.

When a c set is completely described by a suitable binary sequence of the bracket symbols "{" and "}," a q set, eluding complete description like any other q system, is maximally described by a formal sum (superposition) of such sequences. (We revise this notation soon.) Set theory imagines uncountably long sequences of brace symbols generated by transfinite induction; here we assume that only finite sequences have basic physical meaning. Thus q set theory adds to the c set theory of *Principia Mathematica* only the operation of addition representing q superposition. It adds to the usual algebra of cq many-body theory, which already has superposition and much of set theory, only the operation of "bracket," while replacing the usual Grassmann or exterior product by a Clifford product. Finkelstein (1982) and Finkelstein and Rodriguez (1983) introduce the q sets described here. Just as any c object may be modeled as a c set, we suppose any q object may be modeled as a q set.

Notation for q Set Theory. Formally, q set theory is a class $\hat{S}$ provided with one basic object 1, one basic monadic operation of *bracket* $\langle s|$, and three basic dyadic operations of *addition* $s+t$, inner product $s_* t$, and *Clifford product* $s \blacksquare t$, obeying familiar postulates described below.

The needs of the q theory force us to modify the usual notation for c sets. We think of each c set as a predicate or class of a hypothetical object, an "indefinite set" S; as, in fact, a unit class, an atomic predicate, giving complete information about the indefinite set. Even in the c theory we represent these predicates in the q way, by certain submodules called *rays* in a module $\hat{S}$ (generalizing the Hilbert space of the usual q theory and its rays). We call the elements of $\hat{S}$ *plexors*, (Finkelstein, 1972) in anticipation of their use in q topology. The general $\hat{S}$ plexor is defined inductively: 1 is a plexor. If a and b are plexors then so are $b\blacksquare a$, $\langle a|$, and $a + b$. The sum and product here are familiar ways to combine fermi psi vectors and the symbol $\langle a|$ expresses the fundamental topological operation of this theory, called *bracket*. It is used, for example, before composing a simplex $\langle a|\blacksquare\langle b|\blacksquare\langle c|$ from its vertices a, b, and c, and before composing a complex from its simplexes. The bracket symbol $\langle\ |$ combines the brace $\{\ \}$ of Cantor and the ket $|\ \rangle$ of Dirac. We write it also with an operator symbol B:

$$B: u \to Bu = \langle u|$$

Like Cantor's symbol but not Dirac's, brackets nest, as in $\langle\langle a|\blacksquare\langle b||$. Like Dirac's but not Cantor's, brackets add, as in $\langle a| + \langle\langle b||$, so defined that B is linear:

$$\langle u| \pm \langle v| = \langle u \pm v|$$

This is the main difference between the set theory of Takeuti (1979), with its infinity of nontrivial central (superselection, c) quantities, and ours, with none. Any bounded linear function q on $\hat{S}$ with integer values we call a *coplexor*, and we write the value of q on p as qp. We form the coplexor p^* *adjoint* to p (in the sense of Hilbert space) by reading p backwards; this plexor adjoint is inductively defined by the rules $1^* = 1$, $(st)^* = t^*s^*$, $(s + t)^* = s^* + t^*$, and $\langle s|^* = |s^*\langle$. We define the adjoint $^*q$ of a coplexor q analogously.

Classical Plexors. The plexors made inductively from 1 by (Clifford) product and B alone are called *c plexors* because each represents a c set. The set S of c plexors and its interpretation are defined inductively: 1 is in S and represents the null set. If a and b are in S and represent sets A and B, then $\langle a|$ is in S and represents $\{A\}$, and $a\blacksquare b$ is in S and represents the *Boolean union* $A\blacksquare B$ (union less intersection). Classical plexors are simple in the Grassmann sense; the converse is false. By a c basis we mean one consisting of c plexors.

We form the (*plexor*) *transpose* $\sim s$ of a plexor s by first forming the adjoint s^* and then replacing every "$\langle$" in s^* in s^* by a "$|$" and every "$|$"

by a "⟨." The linear operator $\sim$ is an anti-automorphism of $\hat{S}$ and respects bracket: $\sim B = B \sim$. Any c plexor is an eigenplexor of $\sim$; we call the eigenvalue the *signature* of the plexor.

For any plexor p the linear operators of left and right multiplication by p are written $p.$ and $\grave{p}.$ and defined by $p.q = p \blacksquare q$, $p'.q = q \blacksquare p$. Then $(p.)^* = (\sim p)$.

The transpose $\sim$ makes $\hat{S}$ a * algebra of which $\sim$ is the *, and is not to be confused with another anti-automorphism, the *Clifford tranpose*, defined by reversing the monadic factors in an expression for s. The Clifford transpose is called the "main antiautomorphism" by Riesz (1958), and designated by $\tilde{}$ there. Both transposes are diagonal in a c basis, with eigenvalues $+1$ and -1, but they have a different distribution of these eigenvalues. The Clifford transpose is a "surface" reversal and does not respect B, while the plexor tranpose $\tilde{}$ is a total reversal and respects B.

The Clifford Algebra of q Set Theory. $B\hat{S}$, the map of $\hat{S}$ by B, is a submodule $\hat{M}$ of $\hat{S}$. We postulate that $\hat{S}$ is a Clifford algebra over the module $\hat{M} = B\hat{S}$.

If m is any element of $B\hat{S}$ then $m \blacksquare m$ is therefore a c number, which we write as $c(m, m)$, the *Clifford form* of $\hat{S}$.

The Inner Product of $\hat{S}$. To agree with c set algebra we postulate that c plexors belonging to different c sets are orthogonal and have inner product 1 with themselves. (This is a chancy but simplifying assumption. The c sets might, like the c states of the harmonic oscillator, be overcomplete and nonorthogonal.) Any plexor in $\hat{S}$ is a superposition of plexors in S and represents a predicate about a "q set" Ŝ. S is a multiplicative group whose members all have order 2, and $\hat{S}$ is its group algebra.

Closure. A ray in a module M is a special case of a closed submodule of M, which we define first. Let $co\ M$ be the module dual to M, consisting of linear functions on M with c-number values (presently integers). If m is any member of M and m', of $co\ M$, we write

$$m'om$$

for the assertion $m'm = 0$ that the value of m' on m is 0. More generally, we write $L'oL$ for any subclasses L of M and L' of $co\ M$ to mean that $m'om$ holds for all m in L and all m' in L'. We designate by oL the class by all m' obeying $m'oL$, and by $L'o$ the class by all m obeying $L'om$. By the *closure* of any subclass L of M we mean the class $(oL)o$. We call a class *closed* (with respect to o) if it is its own closure. We say x and y are equal *modulo o* and write $x = y \bmod[o]$ if $oxo = oyo$.

Predicate Algebra of *q* Set Theory. In q logic, only closed submodules represent predicates. For example, the submodule of even integers is not a predicate about the q set, while its closure represents the null set (predicate). A *ray* is a submodule that is the closure of one nonzero member. Thus the ray representing the null set consists of the *c* numbers (coefficients) of the module. On the other hand, the submodule representing the universal class (total ignorance about the q set) is the entire module $\hat{S}$. For any class K of plexors we write $[K]$ for the projector upon the closed submodule spanned by K; a closed submodule X is equally defined by $[X]$. Since $[\hat{S}]=1$, the numeral 1 when read as a member of $\hat{S}$ represents the null set and when read as a projector on $\hat{S}$ represents the universal class, a harmless ambiguity. Classical plexors belonging to distinct sets anticommute: $ss' = -s's$; but s and $-s$ represent the same predicate. Thus the c plexors cover the c sets twice.

For simplicity we use integer coefficients in the algebra $\hat{S}$. (It will be seen below, in addition, that some topological information about torsion is lost if we use a field of coefficients, like the rational or complex field, at the start.)

Grassmann Algebra of q Set Theory. Like any Clifford algebra, $\hat{S}$ has a natural Grassman product $s\blacktriangle s'$, also associative, related to the Clifford product ss' by

$$m\blacksquare m' = m\blacktriangle m' + c(m, m')$$

for all monads m and m'. Here $c(m, m')$ is the Clifford form, the polarization of the quadratic form $c(m, m)$ already introduced. The Grassmann product is sometimes regarded as a c limit of the Clifford, in the way that the usual product of c mechanics is the c limit of the noncommutative q product, since in both limits the anticommutator or commutator, respectively, is made to vanish. In S, however, both products have separate and simple interpretations. Just as the Clifford product of the q theory corresponds to the Boolean union of the c, the *q* Grassmann product corresponds to the *c disunion*, defined to be the union in the case of disjoint sets, and undefined (= 0) for nondisjoint sets.

$\hat{S}$ has a Grassmann grade expressing set cardinality, and an element of $\hat{S}$ of grade $0, 1, 2, \ldots$ is called cenadic, monadic, dyadic,.... Plexors of even grade are *bose* and those of odd grade are *fermi*. If an n-adic plexor is the product of n *c* monadics, we call it an n-ad (cenad, monad,...). (C. S. Peirce writes "kenad," but "ceno-" is the more familiar form of the prefix meaning "empty." Moreover it is fitting that c numbers be cenads.) The operator $[G=0]$, which may be read as "the zero-grade part of," stands for the projec(tion opera)tor upon the zero-grade submodule in $\hat{S}$; similarly for $[G=1]$, $[G=2]$,.... As a coordinate of the q set, the grade is then represented in the usual q way by the operator having the n-grade projectors $[G=n]$ for its spectral resolution and the corresponding grades n as eigenvalues:

$$G \coloneqq [G=1]+2[G=2]+3[G=3]+\cdots$$

Any c plexor is an eigenplexor of G and its eigenvalue is its cardinality as a c set. $[G=0]$ is somewhat analogous to the usual vacuum projector, $[G=1]$ to the 1-particle projector,.... But $[G=0]$ produces not the vacuum, which occupies time space, but the null set, which does not: $[G=0]=11^*$. This enables—rather, compels—us to build up the world point by point.

Let $\hat{L}=L(\hat{S})$ be the class of module endomorphisms of $\hat{S}$, "operators." We extract the *grade-n part* of any operator Q of $L(\hat{S})$ by the definition

$$Q[n] \coloneqq [G=n]Q[G=n]$$

The inner product of plexors s with t already defined is given by

$$t_* s=[G=0]((\sim t)s)$$

We use also the operator

$$J \coloneqq [G=0]-[G=1]+[G=2]-[G=3]+\cdots$$

the Gth power of -1, called the "main automorphism" of the Clifford algebra by Riesz (1958). We call J *statistics* because it is $+1$ for bose plexors, -1 for fermi plexors.

Physical Interpretation of q Set Theory. We see that $\hat{S}$ is both a Hilbert space and a Clifford algebra; both structures have immediate physical interpretation.

As Hilbert module, $\hat{S}$ expresses the q aspect of nature. The addition of this module represents q superposition. A transition $s \to t$ is impossible if the inner product or transition amplitude $t_* s$ vanishes.

As Clifford algebra, $\hat{S}$ expresses the combinatory aspect of nature. The Clifford law $m \blacksquare m=1$ for a monadic m expresses what c set theory expresses by the Boole law $s \blacksquare s=0$; indeed, the Clifford product $a \blacksquare b$ represents the Boolean or symmetric sum of sets a and b, the union less the intersection; Boole represented this group operation on c sets with $\oplus$ and we shall henceforth call it the *Boolean union*, writing it as a product. 1 or more generally any nonzero c-number represents the null set and Clifford's law holds in the form

$$m \blacksquare m = c(m, m)$$

for any monadic m; here $c(m, m)$ is a quadratic form in m, the "quadratic Clifford form," first defined for monadics and then extended to all of $\hat{S}$ by the definition

$$c(s, s) = [G = 0](s \blacksquare s)$$

B is linear and preserves inner product and Clifford form.

For example, from $1 \blacksquare 1 = 1$ and the preservation of the Clifford form under B follows $\langle 1| \blacksquare \langle 1| = 1$, $\langle\langle 1|| \blacksquare \langle\langle 1|| = 1, \ldots$. Therefore $\langle 1| \blacksquare \langle\langle 1|| \blacksquare \langle 1| \blacksquare \langle\langle 1|| = -1$; this permits us to express subtraction in terms of addition, multiplication, and bracket alone. At the same time this shows that, while the inner product of $\hat{S}$ is positive definite, the Clifford form of $\hat{S}$ is naturally indefinite. We call plexors of negative square *imaginary*, those of positive square *real*, those of zero square *null*. We assign these plexors the *signatures* $-$, $+$, and 0, respectively. This extends the previously defined concept of signature from c plexors (where it is the eigenvalue of $\sim$) to general plexors (where it is not). We later suppose that the indefinite pseudometric form g of Einstein derives from this indefinite Clifford form, so that $\hat{S}$ carries the seeds of both quantum theory and relativity.

The c-Numbers of q Set Theory. Ultimately we must pass from plexors with integer coefficients to the familiar psi vectors with complex coefficients. We do this in two stages. Here we introduce real coefficients. We regard the reals as limits of rational sequences, introduced to make statistical computations convenient; while a vector with a finite number of rational coefficients expresses the same predicate as any vector with integer coefficients that results from multiplying by a common denominator of the coefficients. It is well known that the system of forbidden transitions (orthogonality relations) determines the transition probabilities (direction cosines) in Hilbert spaces of more than two dimensions. Therefore no additional story is required for the q mechanical transition probability formula

$$\text{prob}(t_* s) = (t_* s)(s_* t)/(t_* t)(s_* s)$$

a rational number.

Eventually we must recover complex coefficients. From the real point of view, i (the square root of -1) is a central (c or "superselection") operator linked with time by Heisenberg's equation of motion, and the minimal (atomic) predicates, one-dimensional projectors of the usual complex Hilbert space, are two-dimensional projectors in the associated real Hilbert space. We therefore postpone i until we have time.

Further Operations of q Set Theory. We define further operations of set theory for plexors first, and we do this by giving their values on a basis for S and then extending to $\hat{S}$ linearly. Then we define the operations for rays projectively.

We *select* a basis thus: By the *binary value* of a sequence of the symbols "⟨," "1," and "|," we shall mean the number 1 if the expression is itself simply "1," and otherwise the number whose binary representation is made by ignoring every "1" in the expression and reading "⟨" as 1, "|" as 0. Thus omitting 1's inside brackets does not affect the binary value of a plexor. By the *select plexor* for a c set we mean that c plexor representing the set with the least binary value. Thus, for example, $\langle 1|\blacksquare\langle\langle 1||$ is a select plexor and $\langle\langle 1|\blacksquare|\langle 1|$ is not. By the select basis for S we mean the select plexors for the c sets arranged in the order of their binary values. This is a largely arbitrary way of singling out a definite basis.

Universal set U. The projector 1 represents the universal class, we have noted. On the other hand there is no universal set in $\hat{S}$. The set U of all finite sets is infinite, hence not represented in S. For each n, however, we may form the product $U(n)$ of the first n monadic plexors in the select basis for S.

$$U(n) \coloneqq \cdots m''m'$$

For any fixed plexor, $U(n)$ will serve as a universal plexor when n is sufficiently large. Accordingly, we shall often write a fictitious plexor U for $U(n)$ in an equation with the understanding that the equation is to hold for sufficiently large n. This U is an effective universal set.

Likewise we define a left complement $U\blacksquare s$ and a right complement $s\blacksquare U$ of any plexor s, with $U\blacksquare s = s\blacksquare U$ mod$[o]$ for classical s. The disunion of plexors s and s' is the Grassmann product $s\blacktriangle s'$ already defined.

Clifford Group. When we use integer coefficients, we require the following generalization of the concept of inverse: We say that s and s', elements of $\hat{S}$, are *projective inverses* of each other, if $s\blacksquare s' = s'\blacksquare s = 1$ mod$[o]$.

The *Clifford group* (Chevalley, 1954) of a Clifford algebra is the group of all elements g with projective inverse g' that map all monads m into monads $g\blacksquare m\blacksquare g'$.

While the Boolean union is invariant under the Clifford group, the disunion is invariant under the general linear group. A bilinear product $s' \cap s'' = s$ is a candidate for a q intersection operation if whenever s' and s'' represent c sets, s represents the c intersection. There exist infinitely many such candidates, and the criteria of group invariance that so narrow the field for the Boolean and disjoint unions do not work here. While the universal plexor U, the Boolean product $s \blacksquare s'$, and the disjoint union $s \blacktriangle s'$ are largely independent of basis, the intersection is very basis dependent. A q union—rather, an infinite number of them—may be similarly defined.

The induced operations on rays in $\hat{S}$ then represent operations on sets (actually, on unit classes of the q set).

Properties of the Bracket *B*. B maps both fermi and bose plexors into fermi. Bracket B and its Hermitian adjoint B^* are related by

$$B^*B = 1$$

$$BB^* = [G = 1].$$

Since B commutes with itself and does not anticommute with itself, B too is bosonic, and the nontrivial commutation relation is

$$B^*B - BB^* = 1 - [G = 1]$$

The maximum number of nested brackets in (an irreducible expression for) a c set is called its *rank*. Rank is defined inductively for c plexors and extended linearly to the rest. Nonzero c numbers have rank 0; the bracket of a plexor of rank r has rank $r+1$; a product of distinct c plexors has the maximum of their ranks. The projector on plexors of rank r is written $[R = r]$, and $R = [R = 1] + 2[R = 2] + \cdots$ defines a q rank, represented by the linear operator R on $\hat{S}$ whose eigenplexors include the c plexors and whose eigenvalues are their ranks. Bracket increases rank by 1, and this leads to the commutation relation

$$RB - BR = B$$

We may give explicit matrices for these operators in the select basis. With respect to the select basis, operators are represented by matrices whose rows and columns are labeled by select c plexors. The operator B is represented by a matrix with one nonzero element in each column, and in a column labeled by the element s of $\hat{S}$ a 1 stands in the row labeled by the monadic $\langle s|$. (If s is select, then so is $\langle s|$.) Of course, R and G are diagonal in the select basis.

Creators and Destructors of q Set Theory. Each monadic m has a crea(tion opera)tor c_m defined as left Grassmann multiplication by m:

$$c_m n := m \blacktriangle n$$

Grassmann multiplication by any monadic m increases grade by 1, leading to the commutation relation

$$Gc_m - c_m G = c_m$$

The Grassmann product $t \blacktriangle s$ may be expressed in terms of the Clifford product:

$$t \blacktriangle s = [t \blacksquare s - J.(s \blacksquare t)]/2$$
$$= [t - Jt`].s/2$$

Dropping the operand s, we have the operator identity

$$c_t = [t - Jt`]/2$$

The adjoint of c_t is the destruct(ion operat)or

$$d_t := c_t^* = [(\sim t) + J(\sim t)`]/2$$

and for monads m, m', the two obey the canonical anticommutation relations:

$$d_m c_{m'} + c_{m'} d_m = m * m'$$

We sometimes call $d_x y$ the derivative of y with respect to x. It is the most natural correspondent within q set theory of the c concept of set inclusion. For c plexors, $d_x y = 0$ means that x is not included in y; and $d_{\langle x|} = 0$ means that x is not a member of y.

We designate by $d_{\hat{S}}$ the *destructor module* consisting of all the destructors d_s for all s in $\hat{S}$.

These operators create and destroy vertices, not particles. The particle creators and destructors depend on the dynamics of the theory, specifically the energy, which has not yet been specified.

Clifford Algebra of Operators. Less than maximal predicates are represented by subspaces of $\hat{S}$, or by projectors on these subspaces. These and coordinates of the q set belong not to $\hat{S}$ but to the algebra $\hat{L} = L(\hat{S})$ of "linear operators" on $\hat{S}$. $\hat{L}$ is also a natural Clifford algebra, over the direct sum of $\hat{M}$ (the monadic submodule) and co$\hat{M}$. Proof: As m ranges over the c monads, the corresponding creators and destructors c_m and d_m range over a set of generators for $\hat{L}$, a count of dimensions verifies. The canonical anticommutation relations for the monadic creators and destructors make $\hat{L}$ a Clifford algebra over the direct sum of their modules, isomorphic, respectively, to $\hat{M}$ and its dual.

Inclusive Statistical Operator. By the *intension* of a c set we mean the membership predicate for the set, or equivalently the class of its members. In many-body theory, the correspondent of intension goes from a many-body or "world" ray projector W to a 1-body statistical operator $w[1]$ that is a projector if and only if the ray of W is simple (contains a product of 1-body psi vectors). In Grassmann algebra, the analogous operation goes from a simple vector to the subspace spanned by its grade-one factors. More generally, any W defines a well-known *inclusive statistical operator* $V = V[W]$ in the many-body fermi space whose n-body part $V[n]$ is the inclusive n-body statistical operator. $V[n]$ is defined by W through the condition that for any n-body quantity $q[n]$, $V[n]$ gives the same expectation value as W:

$$\operatorname{tr}(q[n]V[n]) = \operatorname{tr}(q[n]W)$$

Here the trace operation on the left is over a fixed n bodies, while that on the right covers all the bodies in W. This functional relation from W to V is linear, and may therefore be extended to an arbitrary statistical operator W, called *exclusive* to distinguish it from V. We write this relation as $V = \operatorname{inc}[W]$.

We take this procedure over intact into our Clifford algebra. If w is any plexor, we write inc[w] for the operator in $\hat{L}$ whose grade-n part $v[n]$ obeys the following operation condition for any grade-n operator $q[n]$:

$$\operatorname{tr}(q[n]v[n]) = \operatorname{tr}(wq[n])$$

The operator inc is respected by unitary transformations of $\hat{S}$ that respect grade. For any orthogonal monadics $m', \ldots, m''$ with product plexor $p = m'' \cdots m'$, the grade-1 part of inc(pp^*) is exhibited here:

$$\operatorname{inc}(pp^*) = m'm'^* + \cdots + m''m''^* + \text{high grade terms}$$

It is implied that inc$(11^*) = 0$. Thus the inc of a c plexor is c. $B^*\operatorname{inc}[pp^*]B$ agrees with the c concept of the intension of a finite set p.

Because our objects are finite and some of our classes are not we have the familiar situation that not every class is the intension of a plexor, though every plexor has a class that is its intension. For example the projector 1 is not the intension of any plexor, but only of the fictitious plexor U.

The *inclusion number* for m is the projector

$$N[m] := d_m^* d_m$$

It is the number of m's included in the q set. The *membership number* is the projector $N_{\langle m|}$.

We cannot straightforwardly quantize set equality. Two objects are called equal when they have all their properties in common. But it is impossible to know all the properties of q objects. It is therefore impossible to know the equality of two q objects. Indeed, it is not clear that where equality holds there will be two objects rather than one. We therefore avoid many problems by basing our q set theory on the bracket *operation* rather than on membership and equality *relations*. This has become an important maxim for us: *Operations are more basic than predicates*.

3. QUANTUM TOPOLOGY

We begin with a principle of topological relativity: A q topology will only be defined relative to a q coordinate system CS. For example, time space and energy momentum space have separate topologies, and it is the topology of time space that matters physically, not of energy momentum space. Quanta may be far apart in energy momentum and still interact provided they are close in time space. Thus a q topology breaks unitary symmetry.

Fortunately, the unitary symmetry is already broken in q set theory, which defines its own CS. We shall suppose that some c basis is the CS to use in setting up the topology of time space, and we provisionally use the select basis. For other topologies we would use other bases.

Of the many forms of topology (manifolds, open sets, networks, graphs, etc.), the one most natural for q set theory is simplicial complex topology. Chevalley (1955) recognizes the special relation of simplicial complex topology to Grassmann algebra, which is one aspect of q set theory. He gives every simplicial complex its own Grassmann algebra; we give it its own Clifford algebra as well. Clifford already does this for simplicial complexes in a Euclidean or pseudo-Euclidean space, expressing the topological structure with Grassmann algebra and the metrical with the Clifford form. Our complexes belong to S when they represent the instantaneous q set, and to $\hat{L}$ when they represent the q history of the q set. The Grassmann product underlying the Clifford product of $\hat{L}$ agrees with the Grassmann product ▲

on S and co S, and therefore will be written with the same sign. Under it the creators and destructors in $\hat{L}$ anticommute. Clifford uses the points of a normed linear space to generate his algebra. We use here the algebra generated by the vertices as members of $\hat{S}$ or $\hat{L}$.

Interpretation of q topology. We shall interpret the points of a c simplex as incoherent superpositions or probabilistic mixtures of the vertices. Barycentric coordinates are then probabilities. We define the points of a q simplex as the coherent superpositions of the vertices. The coordinates are then probability amplitudes. In this way, we suppose, each simplex defines it own local linear structure for time space. We take the overall topology of time space as defined by the incidences among these simplices.

In our q topology, a q simplex is formally the same as a q set. The two concepts differ semantically, not syntactically. The term *simplex* specifies a certain topological interpretation. This justifies our use of the term "plexor" for the vectors of $\hat{S}$. We reserve the term "q simplex" (and its synonym, "q set") for the process $\hat{S}$ having plexors for its psi vectors. A plexor is also a simplicial chain interpreted as a psi vector.

$\hat{S}$ may be interpreted as the Hilbert module of a q complex as well as a q set or a q simplex. Just as a set of vertices defines a simplex, a set of simplexes defines a complex. To get to the vertices of the complex we must remove two levels of bracket from a plexor.

We illustrate here the plexor representation of topology. If p, p', p'' are arbitrary plexors, then $\langle p|\blacksquare\langle p'|\blacksquare\langle p''|$ represents a 2-simplex (triangle or triad) with them as vertices, and also a complex with them as simplexes. $\langle\langle p|\blacksquare\langle p'|\blacksquare\langle p''||$ represents a complex with this triangle as its sole simplex, and also a simplex with one vertex. Now we have to distinguish between a vertex p, the simplex $\langle p|$ having p as its sole vertex, and the complex $\langle\langle p||$ having $\langle p|$ as its sole simplex.

The relations complex : simplex : : simplex : vertex : : ... are all the same and we express them all by the bracket of a product, with the possibility of arbitrarily deep infrastructure. This sequence is open-ended; it is just the sequence of sets ordered by the relation of membership. We shall call complex, simplex, vertex, ... by the common name *plexus*. We assign the world—or the plexus under study—the *level* 1, its simplices the level 2, their vertices the level 3, and the new objects of the infrastructure are now called plexi of level 4, 5, ... Rank is a valuation of this order: The rank of a plexus is greater than the rank of any of its members. The grade of a plexus has different interpretations depending on the level of the plexus. For the plexus of first level, grade is a measure, the number of simplicial members; for the second level, the simplices, grade is dimension plus one; and so forth.

$\hat{S}$, as a Clifford algebra of plexors, is not quite the Clifford algebra of forms we would arrive at by generalizing the Grassman algebras associated with a complex by Sorkin (1975) and Weingarten (1976). $\hat{S}$ permits us to multiply two vertices (0-simplexes) of the complex and construct an edge (1-simplex), for example. The addition operation in $\hat{S}$, moreover, is interpreted as q superposition.

Homology and Holonomy. These two topological theories of simplicial complexes figure importantly in physical application: Homology, the theory of the boundary operator d, is important in the study of conserved currents and their sources; its adjoint d^* is an exact finite expression of the field theoretic concept of the exterior derivative. Holonomy, the theory of the covariant derivative D and its integrability, is important in gauge theories, but the theory of D in simplicial complexes is not as developed as the theory of d. We quantize homology only here, and apply the methods we develop to holonomy later.

Two c homologies present themselves for quantization, one with a boundary operator that is multiplicative (the multiplication representing Boolean union), and with a boundary operator that obeys the Leibniz law for a derivative. We call these multiplicative and additive homology, respectively. Since each monad is a square root of a cenad, multiplicative homology is not integral, as is additive homology, but binary. Orientation, important for physics, is lost.

Digression: A multiplicative homology with base other than two is possible, but it means giving up fermi statistics and Clifford algebra. For example, a ternary multiplicative homology seems to require parastatistics, and integral multiplicative homology, Maxwell–Boltzmann statistics. We pursue this though no further. In what follows, homology means additive homology.

The two principals in the c theory are a *boundary operator d* determined by the vertices of the complex, and a *chain module X*, a closed submodule of a Grassmann algebra, which expresses the topology of the particular complex under study (Chevalley, 1955). The stage on which d and X perform is the Grassmann algebra generated by all the vertices of the theory. X is the closed submodule of *chains*, linear combinations of the faces of the complex under study, and the boundary operator d is the sum of the destructors d_v over the vertices:

$$\begin{aligned} d &= d_{v'} + \cdots + d_{v''} \\ &= d_{v' + \cdots + v''} \\ &:= d_v \end{aligned}$$

Here v stands for the sum of all the vertices of the complex. The *homology group* $H(X)$ of a complex with chain module X is the remainder module of the module $[d=0]X$ of *cyclic chains*, (the chains c of zero boundary, $dc=0$) by the module dX of *bounding chains* (the chains b of the form $b=dx$ for some x in X): $H \coloneqq [d=0]X-dX$. Since a Clifford algebra is also a natural Grassmann algebra, this d also acts on any Clifford algebra generated by the p's. The identity $dd=0$ follows from the fact that the d's are anticommuting square roots of 0.

We formulate a q correspondent of this c theory; briefly, we quantize homology.

Quantum Homology. The first ingredient of any homology is its stage, a Clifford algebra. We take this to be $\hat{S}$ at first, for simplicity.

The second ingredient is the boundary operator d. Following the c formula, we first take $d=d_V$, where V is the sum of all vertices, taking this to mean the sum of the unit plexors of the coordinating system CS. This d will be used as the boundary operator for the universal complex U in our first example. The infinite sum is a purely formal intermediary in the calculations, to be handled in the same way as the universal set U. (V actually belongs not to $\hat{S}$ but to $\hat{S}^{**}$.) $V^*V \eqqcolon N$ is the number of points included in V at a given stage of the limiting process. The "plexor" V is the ideal element written as $\rangle$ by Dirac and we call V the *principal plexor* of the CS.

Transformation Properties. By definition, when we change coordinates in $\hat{S}$ the plexors of $\hat{S}$ remain fixed. They are invariant geometric objects. (A *geometric object* is a function assigning a value to each CS, and a rule functionally relating these values is a law of transformation.) Another kind of geometric object is defined by the rule "Take the first basis element of the CS." The value of this object changes with the CS; such objects are called *relative*. The principal plexor V is a relative plexor, not an invariant one. It points along the principal diagonal of the CS. Therefore the boundary operator d is relative, although the plexors it relates can be invariant. We take this to be the relativity of topology that we sought from the start. The crucial question of how d and its plexors transform does not seem to arise in the c theory.

The last ingredient is the chain module X, which must be a closed submodule of $\hat{S}$ specifying the complex under study. X is evidently a q predicate defined by the world or the system under study, expressing empirical facts about the world, laws of nature, or initial conditions, it matters not which. If f is a face of a c complex and v is any point then $d_v f$ is a face of f; the chain module X is required to contain all such faces of its members. Here, therefore, we suppose X is invariant under the destructor module $d_{\hat{S}}$. We assume also that the projector $[X]$ commutes with the grade G.

Consider first the trivial case $X=\hat{S}$ and $[X]=1$. This describes the universal complex U. The operator $d=d_V$ is the boundary operator appropriate to $[X]=1$, and obeys with its adjoint d^* the relations

$$dd^* + d^*d = N[\hat{S}]$$

$$(dd^*)(d^*d) = 0$$

where N is the large number of points in U. Thus aside from normalization, the two terms dd^* and d^*d in the canonical anticommutation relation form a spectral (mutually disjoint, together complete) family of predicates in q logic. $dd^*\hat{S}$ is the space of boundaries in the c theory, so we call its predicate [bounding], and that of $d^*d\hat{S}$, the space of coboundaries, [cobounding], in preparation for the replacement of d by other boundary operators. (Brackets [...] here indicate projectors.) Similarly we define the predicate [cycle] by the subspace of plexors x with $dx=0$: [cycle] $:=$ $[d=0]$. Finally, the q correspondent to the c homology group is the q predicate

$$[\text{hole}] := [\text{cycle}] \text{ and not } [\text{bounding}]$$

"Cycle" also means "without boundary," "boundless": A q hole is the unbounding boundless.

When we use a boundary operator b other than d, we prefix it as in "b cycle."

[Cycle] is equivalent to [not cobounding], since $dx=0$ implies that x is orthogonal to d^*y for all y, and conversely. Thus the completeness of [bounding] and [cobounding] means that [cycle] implies [bounding] and that U has trivial homology: [hole] $=0$.

For a nontrivial homology we consider a world described by a subspace X with $[X]<1$. The boundary operator appropriate to this world is not $d=d_V$ but the derivative with respect to $[X]V$, restricted to $[X]$:

$$b := [X]d[X] = d[X]$$

In the c theory this corresponds to the usual boundary operator d_v for the c complex described by X. Unlike dX, the b-bounding chain module bX need not be closed. A chain x in X might not be a b boundary at the same time that $2x$ is a b boundary. In this case the quotient includes elements of finite order, torsion elements of the homology group, lost if we begin with a field for c numbers. We suppose now that we have computed the torsion and enlarge the c numbers from the integers to the reals. Let $N(b) \coloneqq \text{tr}([X][G=1])$, the number of terms in b.

Let $[X']$ be the orthocomplement of $[X]$ with $[X]+[X']=1$ and $[X][X']=0$, and let z be the transition operator $z \coloneqq [X]d[X']$. Then

$$bb^* + b^*b + zz^* = N(b)[X]$$

$$(bb^*)(b^*b) = 0$$

In general this sum is not a direct sum. The first two terms are orthogonal and express q predicates of [b bounding] and [b cobounding]. The part of $zz^*\hat{S}$ orthogonal both to $bb^*\hat{S}$ and to $b^*b\hat{S}$ means [b cycle and not b bounding], which is the q predicate [b hole] $\coloneqq H$. Because $[X]$ commutes with G, H is the direct sum of its grade-g parts [b homology and $G=g$], and these are the homology groups $H[g] \coloneqq [G=g]H[G=g]$ of the complex.

To sum up, the universal complex U is homologically trivial, but in it are found worlds X that have nontrivial homology. This is our model for the origin of holonomy (and therefore gauge fields) as well: The universal complex is flat, but in it there are curved worlds.

This is an approximate theory, since it rests on a nonmaximal description X of the world. A maximal description of a plexus is a plexor W, the "world plexor." If W is c then $[X] \coloneqq \text{inc}[B^*\text{inc}[WW^*]B]$ may be used as the statistical operator describing the chains of the plexus. (The inner inc in this operator extracts the subcomplexes of the plexus; the inner $B^* \cdots B$ selects the monadic ones and unbrackets their simplicial content; and the outer inc generates the faces of these simplices.)

In an exact q theory following the usual correspondence from cq to c, the q boundary operator, being a function of the world plexus, is represented not by a function of a plexor W but by a linear operator on $\hat{S}$ with c boundary operators for its spectrum. This is an application of noncommutative spectral theory that we leave for later.

Topological Reduction by Measurement. There is a basic discord between the interpretations given by c topology and q logic to the same formulas, and the two theories use different formulas to mean the same thing. It is disconcerting that the boundary of a dyad or line xy is the coherent superposition $x - y$, since the boundary process and the dyad xy are both objects of c topology. If the boundary of xy consists of the points x and y, the q logical projector representing this boundary is not $y - x$ but $(xx^* + yy^*)/2$. If, on the other hand, the boundary is actually a function taking on the values 1 on y and -1 on x, then its q description is the operator $yy^* - xx^*$, not $y - x$.

Quantum topology avoids this discord by taking the formulas of c topology with the interpretations of q logic. We suppose that in fact the boundary of xy at the most fundamental level *is* the coherent superposition $y - x$; that the basic boundary and derivative are q processes like creators and destructors, not c ones nor limits of c ones. The c point-set concept of boundary is related to the basic q boundary by the following reduction process.

In c topology it is meaningful to ask "What is the topology of the world?" (or of the system under study) but this question is not sufficiently leading to be meaningful in a q theory. The most one can ask in q topology is "Which of this complete mutually exclusive set of possibilities is the topology of the world?" The possibilities provided form a CS for $\hat{S}$ (or in general for the relevant Hilbert space), and the operator whose measurement is required by this question we will call $M = M[CS]$. For example, if CS is the select basis, the value produced by M might be the binary value of the plexor representing the world topology.

The usual c boundary is the class of points resulting from applying the q boundary operator and then determining M. For example, the statistical operator $(x - y)(x - y)^*$ is then reduced to one diagonal in the given CS and only the incoherent superposition $xx^* + yy^*$ remains. Thus the reduced boundary is the predicate [one vertex or the other]. This is the class consisting of the two points x and y.

In any CS any plexor w uniquely defines an operator $W = W(M)$, where M is a maximal commuting set whose eigenplexors make up the CS, by the relation $w = WV$, where V is the principal plexor of the CS. W is the psi function for w relative to the CS. While the statistical operator for w is ww^*, the reduced statistical operator is WW^*. In c algebraic topology we use the q w to describe the c WW^*, and we may do this only because we use but one CS, ignoring the relativity of topology. We reluctantly suspend further development of the topological story of the tower of physics to examine now how it supports the higher stories.

4. QUANTUM TIME

We may express time by giving its local invariance group; in general relativity, the action of the Lorentz group on the tangent spaces.

The q simplex defines one local invariance group, the Clifford group G of its vertices. A transformation T in this group, a Clifford transformation, is one of the form

$$T: x \to T(x) = sxs'$$

for some plexor s with projective inverse s', such that T preserves grade and Clifford form. T replaces each vertex by a coherent superposition of them all. The q simplex thus possesses a continuous symmetry that the c simplex lacks. In the terms of the old vector model of the atom, the q simplex points in all directions, and precesses swiftly about any axis. We call a plexor s a *Clifford plexor* if the transformation T it generates in the above way is a Clifford transformation. Over a field of c numbers, the exponential of a dyadic is a Clifford plexor. In the q theory the unitary Clifford transformations generated by imaginary dyadic plexors are particularly important.

Dirac taught us the importance of regarding the Lorentz group as (doubly covered by) a Clifford group, and the Lorentz pseudometric as a Clifford form. Now we hypothesize that the Lorentzian pseudometric form g and the Lorentz group are c limits of the Clifford form and Clifford group of $\hat{S}$. This hypothesis unites the two important Clifford algebras of particle theory, Dirac's Clifford algebra of time-space vectors and the Clifford algebra of Fermi–Dirac "involution" operators. (An involution operator is a sum or difference of creators and destructors, and toggles the quantum into and out of existence.) Points of the world are fermi, their pairs, representing time-space directions, are bose, and the Dirac spin operators of these directions are constructed from bracketed dyads, fermi. To show how neatly the usual spinor concept fits into q simplicial topology, we give it fuller expression in the next paragraphs.

Simplicial Theory of Spinors. Our simplicial spinors, like the geometric spinors of Bank *et al.* (1982), are natural discrete versions of the continuum theory of Clifford numbers and spinors of Riesz 1958 and Kähler 1962, and are close relatives of the lattice fermions of Susskind (1977). In the simplicial theory, however, the spin operators act upon the underlying complex and no spinor fields are postulated, while in the other theories the complex is frozen and the spin operators act upon postulated fermion fields. Let $v, v', \ldots, v''$ be $N+1$ orthogonal monadics in the space $\hat{S}$ of quantum sets. They are then anticommuting square roots of c numbers. In what follows we shall use the v's as (bracketed) vertices of a simplex in N dimensions, and one of them—which we call v—shall serve as origin. At the same time the v's are Clifford numbers, and may be considered as Dirac matrices or their higher-dimensional analogs.

The dyadics formed from the v's other than v are $N(N-1)$ in number and the infinitesimal automorphisms they generate of the N dyadics containing the origin v represent the Lie algebra of an orthogonal group $G(v)$ in N dimensions, with index (number of negative, zero, and positive eigenvalues of the metric) determined by the signatures of the vertices. The typical finite element g of this group maps a dyadic e at the origin into a dyadic f at the origin with $g\blacksquare e = f\blacksquare g$. Since a dyad is here being transformed by multiplication on two sides, it is natural to factor this transformation into two one-sided transformations of the two monads involved. However $g\blacksquare v$ is in general not a monad, and to form a linear space closed under multiplication by g we consider all the simplices that are faces of $v\blacksquare v'\blacksquare \cdots \blacksquare v''$. These fall naturally into two classes with respect to the group $g\ G(v)$: those containing the origin, called "proximal," and those not containing the origin, called "distal." Since g does not contain v, these two classes, each of multiplicity 2^N, are invariant under g. A plexor which transforms by an *initial* factor of g, we call *initial* spinor (relative to the given simplex and origin) and one which transforms with a final factor of the inverse of g, we call a *final* spinor.

Furthermore the group element g is of even grade, and therefore respects the division of the faces of the simplex into odd and even grade. We call spinors of these two classes *odd* and *even*. We thus find eight classes of spinors among the faces of the simplex, each with multiplicity 2^{N-1}. Each spinor may be initial or final, proximal or distal, odd or even.

Since spinors and true plexors transform differently under the orthogonal group $G(v)$, the alignment between them is not invariant. If plexors are invariant geometric objects, spinors are relative ones. In particular it is not quite right to say that a monad is a spin-1/2 object. The spin-1/2 object has components of all grades ranging from 0 or 1 to a maximum by increments of 2. If it includes a monad, it also had triadic, ... components. Rotation tears the topology when it mixes these components.

In this way a spin-1 or vectorial object, a dyad at the origin, is expressed as the product of two spin-1/2 objects, which in one CS are simply its monadic factors or boundary monads.

In three dimensions ($N = 3$) there are four vertices $vv'v''v'''$, and an odd initial spinor has the four components $v, v\blacksquare v'\blacksquare v'', v\blacksquare v''\blacksquare v''', v\blacksquare v'''\blacksquare v'$. These are the four real parts of the usual complex 2-spinor. We identify v', v'', and v''' with the vertices along the x, y, and z axes so that the infinitesimal rotation around the z axis is $v'\blacksquare v''$. The first two spinor components $v, v\blacksquare v'\blacksquare v''$ then have spin up, the last two have spin down, and we recover the complex 2-spinors by the identification

$$v'\blacksquare v''\blacksquare v''' := i$$

Gravity. In choosing the q correspondent of the pseudometric form $g(a, b)$, we should bear in mind the two aspects of $g(a, b)$. Here dyadics a and b correspond to vector fields in the c theory. In the c theory, $g(a, a)$ is the integrated Lorentzian norm of a. In the cq theory, $g(a, a)$ creates and annihilates gravitons with a certain polarization and time space localization specified by a. Thus we expect the q correspondent to be an operator on $\hat{S}$. Insofar as the dyadic a has spin 1, we expect $g(a, a)$ to be atetradic quadratic in a. One natural choice is

$$g(a, b) := \tfrac{1}{2}(a \blacksquare b + b \blacksquare a)$$

Any choice, including this one, amounts to a geometric model of a graviton as a plexic structure composed of the vertices of the world. Here a certain coherent construction of four vertices is proposed.

When the dyads a, b belong to one simplex and a', b' to a disjoint simplex, then $g(a, b)$ commutes with $g(a', b')$. No propagators enter into the commutation relation. This $g(a, b)$ does not correspond to the canonical field variable of the cq theory but to the predynamical third-or hyper-quantized field variable.

While a pentahedron is suggested by the four-dimensionality of time space, the generators of the Dirac group can be represented with the vertices of a tetrahedron. This tetrahedron may be identified with the distal tetrahedron of a pentahedron relative to the origin chosen.

Common time-space experience suggests that the simplexes in our world are mostly pentahedra, four dimensional, with Clifford algebras of real dimension 32, it follows, and that they have indefinite Clifford forms. We call the (Sylvester) index of a quadratic form the sequence of signs of the diagonal elements in its diagonal form, in the order $-, 0, +$. The dominant index of the form g characterizing our time space is $- + + +$ and would arise from simplexes with index $- - + + +$ and $- + + + +$ only. We speculate in Section 7 on why the time space index $- + + +$ so prevails.

Origin of Pseudometric Structure. Finkelstein (1969) identified the time-ordering of events with the order in which their symbols were generated, postulating a basic causal relation. In the present q topology, however, temporality is expressed not by a basic causal relation but by giving each dyad (microscopic direction) a signature, $-$, 0, or $+$; imaginary, null, or real; interpreted as timelike, lightlike, or spacelike. It is simple, also, to present a candidate for CPT (simultaneous charge, space, and time reversal). Because $\sim$ is an all-pervading symmetry, we tentatively interpret $\sim$ as CPT.

In cq theory, temporal order is intimately connected with the order of factors in operator products. If U and V are unitary operators representing two possible evolutions of a q system, then VU represents "U and then V." The product of plexors has no such temporal meaning and is a purely set theoretic or combinatorial operation. As it were, the factors in each plexor reach out to each other and determine their mutual temporal relation regardless of the written order.

A class of transformations of great dynamical importance is the unitary Clifford group, the intersection of the unitary and Clifford groups of $\hat{S}$. Members u of this group connected to the identity are generated by imaginary dyadics a with

$$Ga = 2a,$$
$$\sim a = -a,$$
$$u = \exp(a)$$

We regard this as the plexic analog of a dynamical transformation generated by a vector field, and indeed the dyad a, being a pair of points, is analogous to a vector.

We see here why time-like dyads a may be characterized by their signature, the sign of $g(a, a)$: They would not generate unitary transformations if they had positive signature. This must be the origin of pseudometric structure.

We may now form the one-parameter group $\exp(ta)$ of dynamical plexors. Here first appears a parameter t which we can identify as a time of some kind.

If a has no zero eigenvalues, then the anti-Hermitian dyad a factors uniquely into a Hermitian positive definite plexor h and a unitary anti-Hermitian i commuting strongly with h; that is, i is in the double commutant of h, commuting with every plexor that commutes with h. It is plausible that some such i is the complex i of Heisenberg's equation of motion, but it remains to be seen how far this i can be considered central. Moreover, since this exponential is not time ordered it is not yet in the form corresponding to the dynamics of present q field theory.

The exponential converts the dyad into a superposition of polyads of various (even) grades. This grade corresponds to the number of events in the world (or the system under study) and we will be concerned with its larger values. The prevalence in our ordinary macroscopic experience of pentahedral simplices has yet to be understood and delimited in a basic theory. The world is as if most of the terms in the dyadic a are products of two pentads.

5. THE ONE-STORY TOWER

Let us recapitulate the stories we have quantized so far. Each basic n-ad defines it own linear space and topology by superposition. This gives the world its local topology. The global topology of the world arises from the overlapping of its simplices, another set-theoretic construct. The Lorentzian pseudometric form and its Lorentz group derive from the Clifford form and group of a *q* set. Thus with q sets for story 1, there seems to be no separate story 2 (topology) or 3 (time). We had not anticipated such simplicity. To catch up with our theory, we sharpen the plexus principle:

Story 1 of the q theory is a q topology from which *all* the higher stories of the cq theory may be constructed.

(We have no compelling reason to prefer sets to all other ways of modeling story 1, such as Conway 1976 "games." We use sets because they are well known, built from but two symbols, universal, and have no symmetry.)

In Wheeler's paradoxical terms, the plexus principal requires topology without topology, time without time, and so forth; up to dynamics without dynamics, as in Finkelstein *et al.*

We do not seek a horizontal integration of all fields on story 4 into one, called unification by Einstein and grand unification lately. Such unification is not a sufficiently directive (informative, restrictive) doctrine. We seek a vertical integration of all stories into one; not a hierarchy of theories, but a theory of hierarchy. Since the word "unified" is preempted for a horizontal integration, we call a physical theory with such vertical integration "basic." (Its tower is reduced to its *base*.) An *ultra*basic physics is expressible in terms of a single operator (and the usual operations of q kinematics), like the bracket operator *B* of set theory. No field theory is basic.

6. QUANTUM KINEMATICS

The plexus needs no separate story introducing the fields in the world. We assume as in Section 1 that all fields create and destroy various topological structures, such as defects, dislocations, and kinks, in the plexus. A theory of this kind may still be basic. We conjecture that lines of force exist physically as discrete topological defect lines expressing non-integrable transport of simplices, and charges and other sources are the ends of these lines. This follows discretely the well-known continuum string theories of Dirac, Nambu, Susskind and others.

7. QUANTUM DYNAMICS

We have taken plexors in $\hat{S}$ to describe the physical world and found indications of a basic topology, time, and kinematics. Now we must find a basic dynamics. Where?

We have considered doing without dynamics; perhaps everything is possible, as Peirce, Shestov, and Strindberg, among other antinomians, declare. Wheeler (1980) believes that there is no law, that the universe is higgledy-piggledy. A lawless evolution like Brownian motion, however, increases entropy, and thus disagrees with a unitary evolution defined by a Heisenberg equation of motion, catastrophically as the chronon approaches 0. To save correspondence we must retreat from the anarchy of Finkelstein (1980). We propose instead the renunciation of *absolute* dynamics and the construction of a plurality of *relative* dynamics. For this we require the following extension of the concept of relativity. We shall speak of a trio of relativities, each enlarged by the next, giving an increasingly faithful account of the relations of different CS's:

First (*c*) *relativity*, including Galileo's and Einstein's. It assumes two different CS's have the same possibilities except for a relabelling $x' = f(x)$, a c transformation of the CS.

Second (*cq*) *relativity*, which is Bohr's. It assumes two different CS's see the same system, and that for each CS the possibilities for the system are defined by a complete orthonormal basis in a Hilbert space associated with the system and the CS. It allows each CS to see new possibilities $x' = Ux$ related to the old by a unitary transformation U, a cq transformation of the CS.

Third (*q*) *relativity*. It will allow for the self-evident fact that two different CS's see different worlds, since each sees the other and not itself. A q transformation is generally not a unitary transformation, since the two CS's may have different numbers of possibilities in their basis. The relativity of dynamics is q relativity.

Psi vectors are to abrupt initial or final processes as plexors are to the entire extended process. We follow this analogy further here.

A psi vector is a symbol for a process carried out by CS upon SC at the beginning or end of an experiment.

A psi vector gives more information about CS than about SC. For example, a psi vector given by the spinor $(\cos A, \sin A)$ for a particle of spin-$1/2$ may be used as a symbol for that determination, conveying one bit of information about SC: that it survived the process, one of two possibilities; and infinity bits about CS: that the polarizer has the angle $2A$, one of infinity possibilities. (Let us ignore for just a while longer the actual limits that exist on the information capacity of CS.) We extend the process interpretation from psi vectors to plexors and thus to dynamics in general:

Plexus Principle, continued. A plexor is a symbol for the entire process carried out by CS upon SC.

We need not give a dynamics for a plexor, therefore; the plexor itself gives a dynamics. More generally, a statistical operator on plexors may describe a dynamics nonmaximally.

In standard cq theory, certain processes concentrated in time are represented by initial or final psi vectors; others, distributed over time, by Hamiltonians. We combine the three into one plexor as its initial, medial, and final regions, without sharp interfaces between them in general, since our initial and final processes are not truly instantaneous. Wheeler (1978) emphasizes that we participate in the process that a psi vector describes; but we also participate in the process that a dynamics describes. A cq relativity represents the former participation adequately, but, since the dynamics is described by an absolute geometric object, not the latter. In a q relativity there is no absolute dynamics, only a great many relative dynamics.

Perhaps the transformations of q relativity contain not only c-number parameters of the usual kind but also q numbers, creators, and destructors transforming the old world into the new by creating the part that CS′ sees and CS does not, and destroying what CS sees and CS′ does not. If so, this might provide a physical interpretation for some of the anticommuting *c* numbers that presently seem so mathematically natural in many theories.

At first the conclusion that there are many laws seems opposite to Wheeler's that there are none, but the difference is not that great: Many laws means no Law. The antinomian principle must be annotated: Everything (*in the SC*) is possible (*to a suitably chosen CS*). While it expresses a kind of anarchy from the viewpoint of the SC, from the viewpoint of the CS it expresses holarchy; not in the sense of total determination, which would violate the incompleteness of q logic, but in the sense of determination by the whole. Since we ordinarily consider ourselves a small fraction of the CS, we ordinarily regard neither of these viewpoints as our own.

This plexus principle seems a plausible extension of the usual description of the influence of external currents by c-number fields in the dynamics; it states that the whole dynamics has such a function. What is usually called "the" dynamics for q theory describes the influence of the cosmos at large as well as ourselves on the system. A cq dynamical law typically gives more information about the CS coordinating the experimental process than about the SC. For example, just as a 2-spinor gives one bit about the SC and infinity bits about the CS, a 2×2 unitary matrix representing the dynamics of a spin-1/2 magnetic dipole in a magnetized medium gives 2 bits of information about the evolution of the SC and infinity bits about the medium, part of the CS. Just as the (single-channel) determinations available to each CS are represented by the basic unit psi vectors of the associated frame, those with all vector components 0 except for a 1, the dynamical determinations of any one CS are represented by the basic operators of its frame, those with all matrix elements 0 except for a 1, consisting of a final determination followed by an initial one.

The remarkable success of symmetry principles in discovering useful dynamics must then reflect the approximate symmetry of our actual cosmos at large, and our presently limited ability to disrupt that symmetry.

Finkelstein (1969) attempts to explain the index $- + + +$ of time space by the special role of a two-dimensional complex spinor space in *the* dynamical law. Now that we have no absolute dynamics to blame, we must reverse this: A two-dimensional complex spinor space arises because index $- + + +$ is often the case. Thus we must again seek a reason for the index $- + + +$. Here is the only one we have been able to imagine. We present so thin a conjecture here only because it is demanded and hence to some extent supported by the hypothesis that a basic theory exists.

According to third relativity, the origin of the index of the simplexes in the dynamics of SC must be sought in the CS. We may ask how index propagates from the CS to the SC; this shifts the boundary of the SC outwards. Laws of propagation are dimension-sensitive. For example, the propagator of gravity falls off inversely as the square in the usual three-dimensional space, as the cube in four dimensions, and so on. If index propagates with an exponential dependence upon dimension, then at large distances the lowest dimension that propagates will dominate the higher dimensions. But space dimensions less than three are unviable for various well-known reasons, and may not propagate at all. This allows the usual time space at long distances and more dimensions at shorter ones, as in Kaluza–Klein theories.

8. QUANTUM SEMANTICS

The critical remaining part of a plexus theory is not an absolute dynamics but a semantics, the assignment of meanings to plexors; not another story on the tower, but one of the services, like syntax, that must be provided on every story. What is the plexor (or statistical operator on plexors) representing a particular laboratory or cosmological process? What process is represented by a given plexor? Here is some of what we have constructed so far of plexus semantics.

Each plexor P denotes a dynamical process, what in *cq* theory is represented by an initial psi vector, a sequence of small unitary transformations, and a final psi vector.

Plexor addition represents q superposition of possibilities. Plexor (Clifford) multiplication represents Boolean or symmetric "addition." Bracket represents the formation of a topological or interactive unit (monadic) from constituent processes; expresses interactive links. $P = 1$ stands for the null process.

Each factor of P represents a monadic part (time-space cell with fields) of the process P. The grade of P is its hypervolume in chorons (fundamental q hypervolumes), and its expectation value is a large number. The plexor transpose represents CTP.

Each monadic factor $\langle P'|$ of P defines a simplex whose vertices (bracketed) are the members of P' ("second members" of P). By their overlap the second members of P define the global topology of the world described by P.

The fields usually represented by differential forms of degree n are now represented by $(n+1)$-adic coplexors scalar fields by monadics, covector fields by dyadics, and so forth, subject to membership in P^*. Dually, the fields usually represented by tensors of degree n of the Grassmann algebra over the tangent vectors are represented by $(n+1)$-adic plexors. (Actually, by members of $\hat{S}^{**}$ rather than $\hat{S}$.) Exterior differentiation d is represented by the creator $d_v{}^*$ of the sum v of all the world vertices, restricted to P^*.

The third and further members of P can be varied to provide more possible time-space points. They enter therefore into the time-space coordinates, as well as into fields attached to the points.

9. REMARKS

It is over half a century since Von Neumann (1932) discovered q logic, and a q set theory has been sought for about half that time. In hindsight, one conceptual problem is most responsible for this stasis, besides the usual human ones. It has taken this long to realize how great is the difference between a class, which represents a predicate, and a set, the subject of this predicate. The distinction is troublesome and blurred in mathematics, where all the objects considered—the number 1, the function $\tan x$, and so on—possess, naively speaking, the same degree of actuality. In physics there seems to be a sharp and easy-to-see difference between the set of planets of the solar system and a class of nine possible locations for the planet Earth in the year 1,000,000. In physics, then, it seems understandable to associate the term *classes* with collections of possibilities and *sets* with collections of actualities, a modal distinction Aristotle already points out as necessary for science if not for mathematics.

Von Neumann, for example, made no such sharp distinction in his writings on q logic. He corresponds (Von Neumann, 1954) the q algebra of Hilbert subspaces with both the *c* algebra of sets, incorrectly, and the *c* algebra of predicates or classes, correctly. He had q classes and thought he therefore had q sets, at least at this still formative stage in his theory. We are acutely aware that Von Neumann is as responsible as anyone for making the formal distinction between sets and classes part of modern logic; and that he correctly treats (Von Neumann, 1932) psi vectors as unit classes. But then on the one hand he explicitly states (Von Neumann, 1954): "... the logical treatment corresponds to set theory ... logics correspond to set theory ... ." And on the other hand, in the same paragraph, Von Neumann identifies the measures of sets with probabilities, as if the sets were classes. Evidently it is possible to make the formal distinction between classes and sets without always making the modal one. The confusion of possibility with actuality has haunted quantum theory since Schrödinger interpreted the square of his psi function as charge-density.

Just this conflation of set with class occurs throughout Finkelstein 1969a. Moreover, the vague concept of "flexible logic" proposed there is sharpened in q relativity: Every element of structure of the q logic of SC is only defined relative to CS. How the logic and its Hilbert space depend upon the CS can now be studied quantitatively in simple models at least.

The next most important concept in this work is the bracket operator, which enables us to build the world from the null set instead of from the vacuum. "Why is there something instead of nothing?" Any theory of the present kind has a response to this old question. If "nothing" is represented by the projector $[G=0]$ on the plexor 1, and "something" by its complement $1-[G=0]$, then the *a priori* probability of nothing, which has trace 1, is zero relative to something, which has trace infinity. If by "nothing" is meant 0 instead of 1 (though for us "1" means "nothing," while "0" means nothing), then the probability of nothing is zero all the more. Field theory cannot give this answer because it cannot formulate the question. Instead it answers the question, "Why is time space occupied instead of void?" The projector $[G=0]$ of fermi field theory represents the vacuum in infinite time and space, not the null set.

Here pauses this progress report on what began as a search for the Law of Nature and has become a search for a basic theory. The relativity of dynamical law seems a natural step for what Toulman (1982) has called *postmodern physics*. It would have been an anticlimax were the end of theoretical physics to be some closed formula for a Lagrangian or the like, as unification doctrine suggests. Instead we seek to represent a process greater than and including ourselves, in terms of one basic process; a paradoxical open-ended quest.

ACKNOWLEDGMENTS

A private communication of C. F. von Weizsäcker on the impropriety of considering the universe as a physical system encouraged these considerations. David Bohm long ago emphasized the importance of simplicial homology and the unimportance of absolute law for physics. The willing suspension of disbelief of the participants in the Georgia Tech Quantum Topology Workshop over a number of years was essential for this work. After this paper was

completed we received a private communication from Elihu Lubkin concerning his "physics without time" containing this paragraph:

"I have got bogged down in snarls over selecting a Hamiltonian. Partly I feel it is wrong to *try* to select: A plurality of Hamiltonians is like having external field parameters, and in a thermodynamic context that is provided for by a variety of baths."

This paper is based upon work supported in part by National Science Foundation Grant No. PHY800-7921, and upon the Ph.D. thesis of Rodriguez (1984).

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